simply means that odd ordered variations in the energy, of higher order than two, vanish at $P = P_{cr}$.

How does this affect convergence? To answer this question, consider a (potential energy) functional $K(u)$ expanded about some function v in terms of Gateaux differentials at u_0

$$K(v) = K(u_o) + \delta K(u_o, n) + 1/2!\,\delta^2 K(u_o, n, n) + \ldots \tag{1}$$

Here $n = v - u_o$ and u, v may be vector-valued. Let u_o = exact variational solution of the problem; U = finite-element (Galerkin) approximation of the solution; and $\tilde{U}$ = an arbitrary element in the same subspace of approximants containing U. Suppose that the postbuckling path is stable. Then

$$|K(u_o) - K(U)| \leqq |K(u_o) - K(\tilde{U})| \tag{2}$$

The left-hand side of this inequality is the error in "energy." The right-hand side can be reduced using Eq. (1):

$$K(\tilde{U}) - K(u_o) = \delta K(u_o, E) + 1/2!\,\delta^2 K(u_o, E, E) + 1/3!\,\delta^3 K(u_o, E, E, E) + 1/4!\,\delta^4 K(u_o E, E, E, E) + \ldots$$

where E is the interpolation error

$$E = u_o - \tilde{U}$$

Now, since u_0 is assumed to correspond to a stable critical state, set

$$\delta K(u_o) = 0, \quad \delta^2 K(u_o) = 0, \quad \delta^3 K(u_o) = 0$$

and, for sufficiently small finite-element meshes

$$|K(u_o) - K(\tilde{U})| \leqq C_o \delta^4 K(u_o, E, E, E, E) \leqq C_1 \|E\|^4$$

where C_o and C_1 are constants. Introducing this inequality into Eq. (2) gives

$$|K(u_o) - K(\tilde{U})| \leqq C\|E\|^4$$

For consistent finite-element approximations, $\|E\| \leqq C_o h^r$, where r depends upon the degree of polynomial used in the approximation as well as the degree of the highest derivative in $K(u)$. For example, for piecewise cubics in our beam problem, $r = 4 - 2 = 2$. Notice also for our problem that when $\delta^2 K \approx 0$, $\delta^4 K$ does not depend on quadratic terms in K. We therefore conclude that: 1) the rate-of-convergence at and beyond the critical load is (at least) twice as fast as that before. Moreover, 2) the character of the initial (linear) stiffness does not influence the rate-of-convergence. These conclusions seem to be supported by Cook's experiences.

Finally, turning to item 3 mentioned earlier. It is clear from the abovementioned calculations that had the structure been unstable at the critical load, all of the preceding analysis does not hold. In such cases, the model itself may be unstable. The rate-of-convergence, if any, on points on the postbuckling path will depend upon the global geometry of the structure and its material properties, etc. Conceivably, one might also be able to consider cases in which improved convergence rates were encountered even in cases such as this. The analysis, however, is considerably more involved.

Comment on "Calculation of Turbulent Boundary-Layer Shock-Wave Interaction"

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WILCOX[1] recently reported on a numerical calculation scheme which incorporates a two-parameter turbulence model with finite-difference techniques. Predictions generated by this method, for the two-dimensional viscous-inviscid interaction of an incident oblique shock wave with a turbulent boundary layer, were presented and comparisons with the experimental data of Ref. 2 were shown.†

It should be noted that some of these experimentally determined quantities were incorrectly listed, and/or referred to, in the comparisons made by Wilcox. It is the purpose of this Comment to show the predictions of Ref. 1 in comparison with the combined results of Refs. 2 and 3, and to briefly reiterate the conclusions of this earlier study. At the outset, it should be restated that neither flowfield, as generated by the two distinct shock generator systems of Refs. 2 and 3, was strictly two dimensional; any comparison of these results with the predictions of a mathematically two-dimensional calculation scheme must be viewed with this fact in mind.

Experimentally determined pressure rises to separation and reattachment, upstream influence length, and length of separated flow, as a function of measured over-all pressure rise, are shown in Figs. 1–3; Fig. 4 shows sonic line locations throughout the interaction regions of those four incident shock strength flowfields probed. Measured over-all pressure rises for incipient separation, and experimentally determined separation and reattachment point locations, are also included. Recall that G1 and G2 refer to the two shock generator systems used; G1 was a full-span wedge, while G2 was full span between two side plates, used to cut off the channel sidewall boundary layers. (All nomenclature is as previously listed.)

As stated in Ref. 1, the calculation scheme described did predict the qualitative structure of the interaction flowfield (e.g., existence of both separation and reattachment shocks, a region of reverse flow, and the general shape/behavior of the

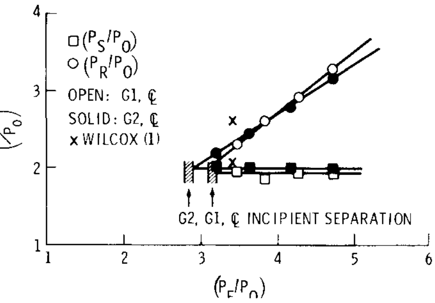

Fig. 1 Pressure rise to separation and reattachment vs over-all pressure rise.

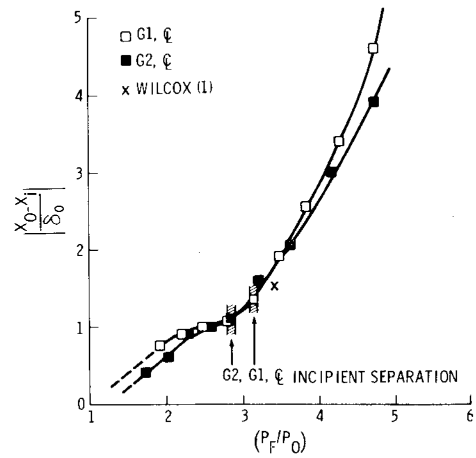

Fig. 2 Upstream influence length vs over-all pressure rise.

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† Experimental results presented in this discussion were generated by the author during his N.R.C. Postdoctoral Research Associateship at NASA Ames Research Center, Moffett Field, Calif.

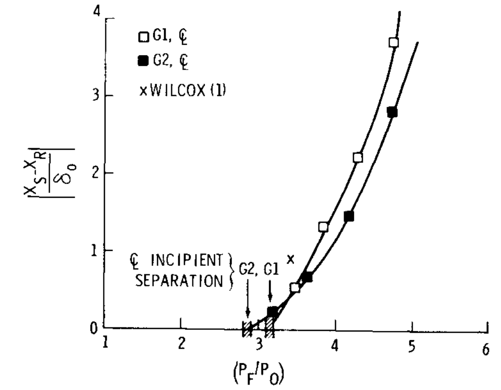

Fig. 3 Length of separated flow vs over-all pressure rise.

sonic line), consistent with experimental observations. The predicted shape of the pressure distribution was also in agreement with experimental observation (i.e., the presence of inflection points and an over-all pressure rise somewhat below that dictated by inviscid theory). In addition, the predicted pressure rise to separation was in close agreement with the experimentally established level ~ 2.0.

With regard to this last point, the free-interaction concept[4] dictates that, for supersonic flow, the pressure rise to separation is independent of the mode of inducing separation; Fig. 1 substantiates this concept (i.e., P_S/P_o independent of P_F/P_o). Flow up to the separation point can be reasoned to be relatively independent not only of the over-all pressure rise and the geometry used to create the viscous-inviscid interaction, but also of any three-dimensional effects which occur downstream of the separation line (the physical location of the separation line, relative to the rest of the flowfield would, however, be sensitive to such three-dimensional effects). The region of flow between the start of the pressure rise and the separation point is physically small ($\sim 1\delta_o$) and is coincident with the very beginning of the adverse pressure gradient. Agreement between predicted and measured P_S/P_o levels, regardless of three-dimensional effects, is thereby expected.

The level of agreement between theory and experiment is reduced when one views those quantities indicative of the over-all scale of the interaction region. The predicted pressure rise to reattachment is greater than measured; the upstream influence length is less than measured, while the predicted length of separated flow is nearly double that measured for an equivalent over-all pressure rise; the predicted sonic line leaves the vicinity of the wall at an angle steeper than observed experimentally; finally, the absolute displacement of the sonic line from the surface is more nearly in agreement with the G1, $\alpha = 11°$, data, rather than

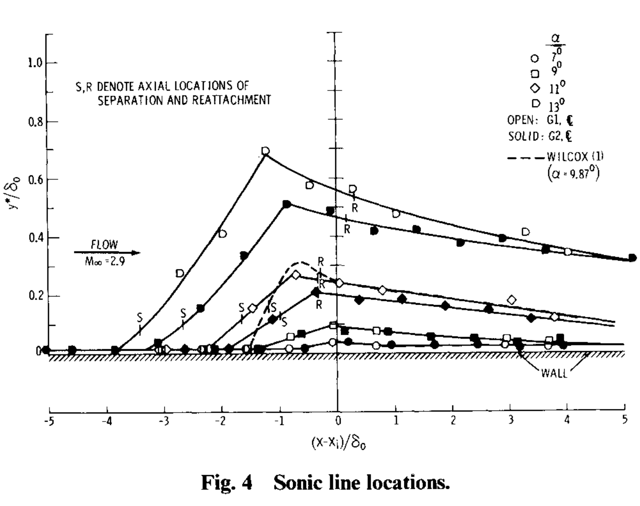

Fig. 4 Sonic line locations.

falling somewhere between the $\alpha = 9°$ and $\alpha = 11°$ data, as might be expected for an $\alpha = 9.87°$ computation.

Combined results of Refs. 2 and 3 have shown that the above-mentioned quantities are dependent on sidewall/corner boundary-layer effects; the extent of influence has been shown to be most prominent for those cases where the test boundary layer is well separated ($|X_S - X_R| > 1\delta_o$).

Exhibited levels of agreement, and disagreement, between the predictions of Ref. 1 and the data of Refs. 2 and 3, do not conclusively prove, or disprove, the merits of such a calculation scheme. It should be noted, however, that the particular shock strength chosen for analysis ($\alpha \approx 10°$) generated, experimentally, only a small region of reverse flow (less than $1\delta_o$), and, accordingly, the interaction flowfield was found to be less removed from a nominally two-dimensional state than those flowfields investigated at higher incident shock strengths. It is felt, therefore, that experimental limitations may not be the total cause for those areas of disagreement noted herein.

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Reply by Author to D. C. Reda

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REDA'S comments pertaining to my listing of his data focus mainly upon size of the separation bubble $(x_R - x_s)/\delta_0$, and the value of the surface pressure at reattachment, p_R/p_0. The source of the discrepancies between the values Reda quotes and those quoted in my Note[1] is a matter of interpretation of the Reda-Murphy data presented in Ref. 2. Specifically, while the curves shown in the abovementioned comment are based upon centerline measurements, I have chosen to include both centerline and off-centerline data in my comparisons to provide a measure of experimental data scatter. The computed surface pressure distribution is shown in Fig. 1; open symbols denote measured pressures at separation and reattachment. Surface pressure for the Reda-Murphy $G1$, $\alpha = 10°$ flowfield indicates that $p_R/p_0 \simeq 2.31$ on the tunnel centerline while p_R/p_0 is as high as 2.50 away from the centerline. The calculated value of p_R/p_0 of 2.63 differs, as quoted in the Note, by 5% from the off-centerline value. Similarly, I interpreted the extent of the separated region as lying between the separation point farthest upstream and the reattachment point farthest downstream. With this interpretation, the length of the Reda-Murphy separated region is $(x_R - x_s)/\delta_0 \simeq 0.86$ compared to 0.91 in the numerical flowfield.

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